J. Math. Kyoto Univ. (JMKYAZ)
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# An approach to the Biharmonic pseudo process by a Random walk

By

Sadao Sato*

## 1. Introduction

We consider the partial differential equation

$$\frac{\partial u}{\partial t} = -\frac{1}{8}\frac{\partial^4 u}{\partial x^4} \tag{1.1}$$

Here $\Delta^2 = \partial^4/\partial x^4$ is called the biharmonic operator. The fundamental solution of this equation is

$$q(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, e^{-ix\lambda - \frac{1}{8}\lambda^4 t} \tag{1.2}$$

As shown by Hochberg[H], this function is not positive valued. Hence the usual probabilistic method is not applicable. However, there are several probabilistic approaches to this equation.

Krylov[K],later Hochberg[H] and Nishioka[N1] considered a signed finitely additive measure on a path space, which is essentially the limit of a Markov chain with the signed distribution $q(t,x)$. Krylov obtained a path continuity of the *biharmonic pseudo process*. Note that there does not exist the $\sigma$-additive measure on a path space realizing a stochastic process related to the biharmonic operator. We can consider the biharmonic pseudo process only through a limit procedure from a finite additive measure. Our aim of this paper is to propose a new limit procedure for the biharmonic pseudo process.

Nishioka considered the first hitting time to $(-\infty,0)$ and obtained the joint distribution of the first hitting time and the first hitting place. His remarkable result is that there appears the differentiation at 0 for the distribution of the first hitting place(see

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*Tokyo Denki University

section 6). However, since the jump distribution $q(t,x)$ is spread on the entire of $\mathbf{R}$, it is difficult to understand this fact intuitively.

In this paper, we define a random walk , which has the only finite jumps with signed measure. And we will show Nishioka's result by a scaling limit. We can get short and elementary proofs for his result. And we can understand that the differentiation at 0 is caused by the hitting measures on the two boundary points, which are infinite as a limit and have opposite signs (see Proposition 5.1). We can also get the distribution of the first exit time and place for finite intervals(section 8).

## 2. Definition of a biharmonic random walk

Let $(Y_i)$ be the i.i.d. random variables with the *signed* distibution defined by

$$P(Y=0)=p, P(Y=\pm 1)=q, P(Y=\pm 2)=r, \tag{2.1}$$

where each of probabilities should be prescribed later. Consider the random walk

$$X_n = X_0 + Y_1 + \cdots + Y_n.$$

Let $f$ be a measurable function and we shall write

$$E[f(X_n)] = \sum_k f(k)P[X_n = k]$$

and

$$E_x[f(X_n)] = E[f(X_n)|X_0 = x].$$

Also we shall write

$$P_x[X_n = k] = P[X_n = k|X_0 = x].$$

By the Taylor expansion, we have

$$E_x[f(X_1)] - f(x) = (p+2q+2r-1)f(x) + (q+4r)f''(x) + (\frac{q}{12} + \frac{4r}{3})f''''(x) + \cdots .$$

If we set

$$p+2q+2r=1, \quad q+4r=0,$$

that is

$$p=1+6r, \quad q=-4r,$$

then we have

$$E_x[f(X_1)] - f(x) \simeq r\Delta^2 f(x). \tag{2.2}$$

Here, number $r$ must be negative, since we match (2.2) to (1.1) . Now we compute the characteristic function of $Y$.

$$\begin{aligned} M(\mu) &\stackrel{\text{def}}{=} E(e^{i\mu Y}) = p + 2q\cos\mu + 2r\cos 2\mu \\ &= 4r(1-\cos\mu)^2 + 1. \end{aligned}$$

Since $r$ is negative, we get $\max_\mu M(\mu) = 1$ and $\min_\mu M(\mu) = 16r+1$. Let $r = -\frac{1}{8}$, and we have $-1 \le M(\mu) \le 1$. Since

$$E_x(e^{i\mu X_n}) = e^{i\mu x} M(\mu)^n,$$

this quantity converges to zero as $n \to \infty$ if not $|\cos\mu| = 1$. Now we have

$$p = \frac{1}{4}, \quad q = \frac{1}{2}, \quad M(\mu) = 1 - \frac{1}{2}(1-\cos\mu)^2.$$

Let $\{\mathscr{F}_n\}_{n\ge 0}$ be the filtration generated by $\{X_n\}_{n\ge 0}$. Since

$$p + 2q + 2|r| = \frac{3}{2},$$

the total variation of the measure restricted on $\mathscr{F}_n$ is equal to $(\frac{3}{2})^n$. Note that $\mathscr{F}_n$ is a finite set, which consists of $5^n$ atoms, and the mean $E_x[\cdot]$ is defined for any event of $\mathscr{F}_n$. Set

$$p(n,k) = P_0[X_n = k].$$

Then we have

$$\sum_k p(n,k) e^{ik\mu} = M(\mu)^n.$$

Thus $p(n,k)$ is the Fourier cofficient of the right-hand . Therefore

$$p(n,k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} M(\mu)^n e^{-ik\mu} d\mu \tag{2.3}$$

and we have

$$\lim_{n\to\infty} p(n,k) = 0. \tag{2.4}$$

**Remark 1.** We can take any r in$[-\frac{1}{8}, 0)$for our purpose. For example, if we take $r = -\frac{1}{16}$, then we have $0 \le M \le 1$ and this situation looks like better for some cases. However, we point out that the value $-\frac{1}{8}$ seems to be best for the numerical simulation to be rapid.

## 3. Scaling limit to continuous time and space

For $\varepsilon > 0$, set

$$x = k\varepsilon, \quad t = n\varepsilon^4.$$

Then we have

$$\begin{aligned} p_\varepsilon(t,x) &\overset{\text{def}}{=} p(n,k) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} M(\mu)^{t/\varepsilon^4} e^{-ix\mu/\varepsilon} d\mu \\ &= \frac{1}{2\pi} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} M(\varepsilon y)^{t/\varepsilon^4} e^{-ixy} dy \cdot \varepsilon. \end{aligned}$$

We will compute the following limit

$$q(t,x) \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} p_\varepsilon(t,x).$$

Let $\delta(x)$ be Dirac's delta function and we clearly have

$$q(0,x) = \delta(x).$$

Since $M(\varepsilon y)$ is an even function, we consider

$$\begin{aligned} \frac{1}{\varepsilon} p_\varepsilon(t,x) &= \frac{1}{\pi}\Big(\int_{\pi/\varepsilon-\pi/\sqrt{\varepsilon}}^{\pi/\varepsilon} + \int_{\pi/\varepsilon^{1/4}}^{\pi/\varepsilon-\pi/\sqrt{\varepsilon}} + \int_0^{\pi/\varepsilon^{1/4}}\Big) M(\varepsilon y)^{t/\varepsilon^4} \cos(xy)\, dy \\ &= I_1 + I_2 + I_3 \quad (say). \end{aligned}$$

First, we get

$$\begin{aligned} |I_1| &\le \frac{1}{\pi}\int_{\pi/\varepsilon-\pi/\sqrt{\varepsilon}}^{\pi/\varepsilon} |1 - \frac{1}{2}(1-\cos\varepsilon y)^2|^{t/\varepsilon^4} dy \\ &= \frac{1}{\pi}\int_0^{\pi/\sqrt{\varepsilon}} (-1 + \frac{1}{2}(1+\cos\varepsilon u)^2)^{t/\varepsilon^4} du \\ &\approx \frac{1}{\pi}\int_0^{\pi/\sqrt{\varepsilon}} (1-\varepsilon^2 u^2)^{t/\varepsilon^4} du \\ &\approx \frac{1}{\pi}\int_0^{\pi/\sqrt{\varepsilon}} e^{-u^2 t/\varepsilon^2} du \to 0 \quad (\varepsilon \to 0). \end{aligned}$$

Second we estimate $I_2$. Note that $M(\mu)$ is monotone in the interval $[0,\pi]$ and

$$M(\pi - \pi\sqrt{\varepsilon}) \approx -1 + \varepsilon\pi^2, \quad M(\pi\varepsilon^{3/4}) \approx 1 - \frac{1}{8}\varepsilon^{3/2}\pi^2.$$

Therefore

$$|I_2| \le \frac{\pi}{\varepsilon}(1 - \frac{1}{8}\varepsilon^{3/2}\pi^2)^{t/\varepsilon^4} \approx \frac{\pi}{\varepsilon} e^{-\pi^2 t/8\varepsilon^{5/2}} \to 0.$$

Finally, in the interval of the integral $I_3$, we have

$$M(\varepsilon y) \approx 1 - \frac{1}{8}\varepsilon^4 y^4.$$

Now we obtain the following conclusion:

**Theorem 3.1.** *By the scaling $x = k\varepsilon, t = n\varepsilon^4$, we have*

$$\begin{aligned} q(t,x) &\stackrel{\text{def}}{=} \lim_{n\to\infty} \frac{1}{\varepsilon} p(n,k) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{1}{8}y^4 t - ixy} dy, \end{aligned}$$

*which is the fundamental solution of the equation*

$$\frac{\partial u}{\partial t} = -\frac{1}{8}\frac{\partial^4 u}{\partial x^4}.$$

Next we consider the initial value problem for (1.1).

**Theorem 3.2.** *Let $f$ be a continuous function in $L^2(R)$. Suppose that $f$ is the Fourier transform of a function $\hat{f}$ in $L^1(R)$ , that is*

$$f(x) = \int_{-\infty}^{\infty} e^{-i\lambda x} \hat{f}(\lambda) d\lambda. \tag{3.1}$$

*Set*

$$u_\varepsilon(t,x) \stackrel{\text{def}}{=} E_{[\frac{x}{\varepsilon}]}(f(\varepsilon X_{[\frac{t}{\varepsilon^4}]})), \tag{3.2}$$

*where $[\cdot]$ denotes its integer part. Then there exists the limit*

$$u(t,x) = \lim_{\varepsilon \to 0} u_\varepsilon(t,x)$$

*which satisfies (1.1) and*

$$u(0,x) = f(x).$$

*Proof.* Let

$$l = [x/\varepsilon], \quad n = [t/\varepsilon^4].$$

Noting that $E_l[\cdot]$ is the measure on a finite set, we have

$$\begin{aligned} u_\varepsilon(t,x) &= E_l[\int_{-\infty}^{\infty} e^{-i\lambda\varepsilon X_n} \hat{f}(\lambda) d\lambda] \\ &= \int_{-\infty}^{\infty} d\lambda\, \hat{f}(\lambda) \sum_k e^{-i\lambda\varepsilon(l+k)} p(n,k) \\ &= \int_{-\infty}^{\infty} d\lambda\, \hat{f}(\lambda) e^{-i\lambda\varepsilon l} M(\varepsilon\lambda)^n. \end{aligned}$$

Since $\hat{f}$ is in $L^1(R)$ and $M(\varepsilon\lambda)^n$ is bounded, Lebesgue's dominated convergence theorem implies

$$\begin{aligned} u(t,x) &= \int_{-\infty}^{\infty} d\lambda\, \hat{f}(\lambda) e^{-i\lambda x} e^{-\frac{1}{8}\lambda^4 t} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dy f(y) \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda(x-y) - \frac{1}{8}\lambda^4 t}, \end{aligned}$$

which completes the proof. □

## 4. Hitting measure to $(-\infty, 0)$

For each $|s| < 1$, the Green operator of $\{X_n\}_{n \geq 0}$ is defined by

$$G_s f(l) = \sum_{n=0}^{\infty} s^n E_l[f(X_n)].$$

By the Parseval's equality, we have

$$\sum_k p(n,k)^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} M(\mu)^{2n} d\mu = p(2n,0) \leq 1$$

which is monotone decreasing to zero as $n$ goes to $\infty$. Schwarz's inequality implies

$$|E_l[f(X_n)]| = |\sum_k p(n,k-l)f(k)| \leq \sqrt{p(2n,0)} \parallel f \parallel_{\ell^2},$$

and we obtain the following:

**Proposition 4.1.** *For every $f$ of $\ell_2$, $G_s f$ is analytic in $|s|<1$, and it holds that*

$$|G_s f(l)| \leq \frac{1}{1-|s|} \parallel f \parallel_{\ell^2}.$$

We define the first hitting time to the set $Z \cap (-\infty, 0)$ as

$$\sigma = min\{n; X_n < 0\}$$

and we shall write

$$\tilde{p}(n,l,k) \stackrel{\text{def}}{=} P_l[X_n = k; n < \sigma],$$

since $\mathscr{F}_n$ is a finite set, here the right-hand quantity is well-defined. Then we have

$$\begin{aligned}\tilde{p}(n,l,k) &= P_l[X_n = k] - P_l[X_n = k; \sigma \leq n] \\ &= P_l[X_n = k] - \sum_{i=-1,-2} \sum_{m=0}^{n} P_l[\sigma = m, X_\sigma = i] P_i[X_{n-m} = k],\end{aligned}$$

where the last equality is implied from the strong Markov property of $\{X_n\}$. For $l$ and $k$ in $Z$, define

$$g_s(l,k) = G_s 1_k(l),$$

$$\tilde{g}_s(l,k) = \sum_{n=0}^{\infty} s^n \tilde{p}(n,l,k),$$

$$H_i(s,l) = \sum_{n=0}^{\infty} s^n P_l[\sigma = n, X_\sigma = i].$$

Note that $\tilde{g}_s(l,k)$ and $H_i(s,l)$ are analytic in $|s| < \frac{2}{3}$, since the total variation of $E_x[\cdot]$ restricted on $\mathscr{F}_n$ is equal to $(\frac{3}{2})^n$. From the definition, we get

$$\tilde{g}_s(l,k) = g_s(l,k) - \sum_{i=-1,-2} H_i(s,l) g_s(i,k). \tag{4.1}$$

Fix $|s|<1$, and we shall use the notation

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{in\mu}}{1-sM(\mu)} d\mu \quad (= c_{-n}). \tag{4.2}$$

Since (2.1) and (4.2) imply that

$$g_s(l,k) = c_{k-l}, \tag{4.3}$$

we have

$$\tilde{g}_s(l,k) = c_{k-l} - \sum_{i=-1,-2} H_i(s,l) c_{k-i}. \tag{4.4}$$

Especiallly, when $k = -1$ and $-2$, we have

$$0 = c_{l+1} - c_0 H_{-1}(s,l) - c_1 H_{-2}(s,l), \tag{4.5}$$

$$0 = c_{l+2} - c_1 H_{-1}(s,l) - c_0 H_{-2}(s,l). \tag{4.6}$$

**Proposition 4.2.** *For* $|s| < 1$ *and* $l \in Z \cap [0,\infty)$,

$$H_{-1}(s,l) = \frac{c_{l+1}c_0 - c_{l+2}c_1}{c_0^2 - c_1^2}, \tag{4.7}$$

$$H_{-2}(s,l) = \frac{c_{l+2}c_0 - c_{l+1}c_1}{c_0^2 - c_1^2}. \tag{4.8}$$

*Moreover* $H_i(s,l)$ *are analytic in* $|s| < 1$.

*Proof.* We know that $c_l$'s are analytic in $|s| < 1$. The first part of the proposition is clear from (4.5) and (4.6), if we can show that the denominator $c_0^2 - c_1^2$ has no zero in $|s| < 1$. We have

$$\begin{aligned} c_0 - c_1 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1-\cos\mu}{1 - sM(\mu)} d\mu \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-\cos\mu)(1-\bar{s}M(\mu))}{|1-sM(\mu)|^2} d\mu. \end{aligned}$$

So, we get

$$Re(c_0 - c_1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1-\cos\mu)(1-Re(s)M(\mu))}{|1-sM(\mu)|^2} d\mu > 0.$$

Similarly, we obtain $Re(c_0 + c_1) > 0$. Now the second part is immediate and the proof is complete. □

## 5. Existence of the Hitting measure

In this section, we will compute the limit

$$\lim_{s \to 1} H_i(s,l)$$

which is the interpretation of the hitting measure

$$\sum_{m=0}^{\infty} P_l[\sigma = m, X_\sigma = i] = P_l[X_\sigma = i, \sigma < \infty].$$

However, we note that the latter quantity may not be convergent, because these terms are signed.

At first, we study the $c_n$'s in (4.2). Set $z = e^{i\mu}$, and change a variable. The residue theorem implies

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{in\mu}}{1-s\{1-\frac{1}{2}(1-\cos\mu)^2\}}d\mu && (5.1)\\ &= \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{dz}{z}\frac{z^n}{1-s\{1-\frac{1}{2}(1-\frac{z^2+1}{2z})^2\}} && (5.2)\\ &= \frac{1}{2\pi i s}\int_{\mathscr{C}} dz \frac{8z^{n+1}}{8(\frac{1}{s}-1)z^2+(z-1)^4} && (5.3)\\ &= \frac{8}{s}\sum_{|z|<1} Res(\frac{z^{n+1}}{4v^4z^2+(z-1)^4}) && (5.4)\end{aligned}$$

where $\mathscr{C}$ is the unit circle in the complex plain and

$$v = (2(\frac{1}{s}-1))^{1/4}. \tag{5.5}$$

In the remainder of this section, we restrict $s$ as a real number. So, we assume that $0 < s < 1$ and $v$ is positive. We consider zeros of the denominator $f(z) = 4v^4z^2 + (z-1)^4$. By an elementary calculation, we obtain that

$$\alpha = 1 + iv^2 - v\sqrt{\frac{\sqrt{v^4+4}-v^2}{2}} - iv\sqrt{\frac{\sqrt{v^4+4}+v^2}{2}} \tag{5.6}$$

is a zero point, which satisfies $|\alpha| < 1$. We note that $\alpha$ satisfies

$$(1-\alpha)^2 = 2iv^2\alpha. \tag{5.7}$$

And $\bar{\alpha}$ is another zero point inside of $\mathscr{C}$. Other two zeros are outside of $\mathscr{C}$. Therefore

$$\begin{aligned} c_n &= \frac{8}{s}(\frac{\alpha^{n+1}}{f'(\alpha)} + \frac{\bar{\alpha}^{n+1}}{f'(\bar{\alpha})})\\ &= \frac{16}{s}Re(\frac{\alpha^{n+1}(\alpha-1)}{8v^4\alpha(\alpha-1)+4(\alpha-1)^4}) && (5.8)\\ &= \frac{2}{sv^4}Re(\frac{\alpha^n(1-\alpha)}{1+\alpha}).\end{aligned}$$

In the following, we shall use

$$\tilde{c}_n = Re(\frac{\alpha^n(1-\alpha)}{1+\alpha}) = \frac{sv^4c_n}{2} \tag{5.9}$$

instead of $c_n$ for simple notations. Note that $v$ tends to zero as $s$ tends to 1. Let $v$ be small and Taylor expansion derives that

$$\alpha \approx 1-(1+i)v+iv^2+\frac{1}{4}(1-i)v^3-\frac{1}{32}(1+i)v^5, \tag{5.10}$$

$$\log(\alpha) \approx -(1+i)v-\frac{1}{12}(1-i)v^3-\frac{3}{160}(1+i)v^5, \tag{5.11}$$

$$\frac{1}{1+\alpha} \approx \frac{1}{2}+\frac{(1+i)}{4}v+\frac{1}{16}(1-i)v^3-\frac{3}{128}(1+i)v^5, \tag{5.12}$$

$$\frac{\alpha}{1+\alpha} \approx \frac{1}{2}-\frac{(1+i)}{4}v-\frac{1}{16}(1-i)v^3+\frac{3}{128}(1+i)v^5, \tag{5.13}$$

$$\alpha^n \approx 1-n(1+i)v+in^2v^2+\frac{n(4n^2-1)}{12}(1-i)v^3, \tag{5.14}$$

$$\tilde{c}_n \approx v(\frac{1}{2}+\frac{1-4n^2}{8}v^2+\frac{n(n^2-1)}{3}v^3). \tag{5.15}$$

From (5.7) and (5.9), we obtain the following simple equalities

$$\tilde{c}_n+\tilde{c}_{n+1} = Re(\alpha^n(1-\alpha)), \tag{5.16}$$

$$\tilde{c}_n-\tilde{c}_{n+1} = Re(\frac{\alpha^n(1-\alpha)^2}{1+\alpha}) \tag{5.17}$$

$$= -2v^2 Im(\frac{\alpha^{n+1}}{1+\alpha}) \tag{5.18}$$

which are often used in our computations. By Proposition 4.2 and the aboves, we get

$$\begin{aligned} H_{-1}(s,l)+H_{-2}(s,l) &= \frac{c_{l+1}+c_{l+2}}{c_0+c_1} = \frac{\tilde{c}_{l+1}+\tilde{c}_{l+2}}{\tilde{c}_0+\tilde{c}_1} \\ &= \frac{Re(\alpha^{l+1}(1-\alpha))}{Re(1-\alpha)} \\ &\approx \frac{v+(-l^2-3l-9/4)v^3}{v-1/4v^3} \to 1 \quad as \quad s\to 1, \end{aligned}$$

and

$$\begin{aligned} H_{-1}(s,l)-H_{-2}(s,l) &= \frac{c_{l+1}-c_{l+2}}{c_0-c_1} = \frac{\tilde{c}_{l+1}-\tilde{c}_{l+2}}{\tilde{c}_0-\tilde{c}_1} \\ &= Im(\frac{\alpha^{l+2}}{1+\alpha})/Im(\frac{\alpha}{1+\alpha}) \\ &\approx \frac{(l+\frac{3}{2})v^3+(-l^2-3l-2)v^4}{\frac{1}{2}v^3+\frac{1}{8}v^5} \to 2l+3 \quad as \quad s\to 1. \end{aligned}$$

Thus we get the following proposition:

**Proposition 5.1.** *For $l \in Z\cap[0,\infty)$,*

$$H_{-1}(1,l) = 2+l, \tag{5.19}$$

$$H_{-2}(1,l) = -1-l. \tag{5.20}$$

**Corollary 5.1.** *As $v \to 0$, the followings hold:*

$$g_s(l,k) \approx \frac{1}{sv^3}, \tag{5.21}$$

$$\tilde{g}_1(l,k) = \begin{cases} 8 + \frac{28}{3}l + 8k + 12kl + 4kl^2 - \frac{4}{3}l^3 & (k \geq l) \\ 8 + \frac{28}{3}k + 8l + 12kl + 4lk^2 - \frac{4}{3}k^3 & (k \leq l) \end{cases} . \tag{5.22}$$

*Proof.* (5.21) is easily obtained from (4.3) and (5.8). The latter is obtained by an elementary but long calculation. We will omit the details. □

Now we can explain why the differentiation appears at 0. Let $f(x)$ be a differentiable function and consider

$$E_l[s^\sigma f(\varepsilon X_\sigma)] = f(-2\varepsilon)H_{-2}(s,l) + f(-\varepsilon)H_{-1}(s,l)$$
$$= \frac{f(-\varepsilon)+f(-2\varepsilon)}{2}(H_{-1}(s,l)+H_{-2}(s,l)) + \frac{f(-\varepsilon)-f(-2\varepsilon)}{2}(H_{-1}(s,l)-H_{-2}(s,l)).$$

By letting $s \to 1$, we get

$$\lim_{s\to 1} E_l[s^\sigma f(\varepsilon X_\sigma)] = \frac{f(-\varepsilon)+f(-2\varepsilon)}{2} + \frac{f(-\varepsilon)-f(-2\varepsilon)}{2}(2l+3).$$

We take the same scaling as in Theorem 3.1, that is, set $x = l\varepsilon$ and fix it. As $\varepsilon$ goes to zero, the right hand side of this equation goes to

$$f(0) + f'(0)x.$$

Therefore we obtain

$$Q_x[f(Z_\sigma)] = f(0) + f'(0)x$$

for the *biharmonic pseudo process* $\{Z_t\}$, that is the "stochastic process" obtained through the scaling limit in Theorem 3.1. Here, $Q_x[\cdot]$ denotes the expectation for $\{Z_t\}$ starting from $x$. However, there does not exist such measure in the usual sense. We will consider $Q_x[\cdot]$ only through a limit procedure for our random walk $\{X_n\}$. In this discussion, we used double limit procedure. We will give more direct proof by using the scaling limit in the next section.

## 6. Scaling limit of the first hitting time and place

We will compute the Fourier-Laplace transform of the first hitting place and time:

$$Q_x[e^{i\mu Z_\sigma - \lambda\sigma}],$$

which was first obtained by Nishioka[N1]. We use the same scaling as in section 3. Let $f(x)$ be a bounded $C^1$-function and $x = l\varepsilon$. We consider

$$Q_x[e^{-\lambda\sigma} f(Z_\sigma)] \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0} E_l[e^{-\varepsilon^4\lambda\sigma} f(\varepsilon X_\sigma)], \tag{6.1}$$

where $x = l\varepsilon$ is fixed. The left-hand side of this equation has only symbolical meaning. Here

$$\sigma = \inf\{t > 0; Z_t < 0\}$$

is the first hitting time to the set $(-\infty, 0)$ of the biharmonic pseudo process $\{Z_t\}$. We consider the mean $Q_x[\cdot]$ for $\{Z_t\}$ through the scaling limit of $\{X_n\}$. Now we compute the right-hand side of (6.1):

$$\begin{aligned}
&E_l[e^{-\varepsilon^4\lambda\sigma} f(\varepsilon X_\sigma)] \\
&= f(-\varepsilon)E_l[e^{-\varepsilon^4\lambda\sigma}; X_\sigma = -1] + f(-2\varepsilon)E_l[e^{-\varepsilon^4\lambda\sigma}; X_\sigma = -2] \\
&= f(-\varepsilon)H_{-1}(e^{-\varepsilon^4\lambda}, l) + f(-2\varepsilon)H_{-2}(e^{-\varepsilon^4\lambda}, l) \\
&= \frac{1}{2}(f(-\varepsilon) + f(-2\varepsilon))(H_{-1} + H_{-2}) + \frac{1}{2}(f(-\varepsilon) - f(-2\varepsilon))(H_{-1} - H_{-2}) \\
&= I_1 + I_2 \quad (say).
\end{aligned}$$

Since $s = e^{-\varepsilon^4\lambda}$ , we have

$$v = (2(e^{\varepsilon^4\lambda} - 1))^{1/4} \approx \varepsilon(2\lambda)^{1/4}.$$

For the simplicity, we set

$$\nu = (2\lambda)^{1/4}. \tag{6.2}$$

By (5.11), we have

$$\begin{aligned}
\alpha^l &= \exp(l(-v(1+i) + O(v^2))) &(6.3)\\
&\approx \exp(-l\varepsilon\nu(1+i) + O(\varepsilon)) &(6.4)\\
&\to e^{-\nu x(1+i)} \quad as \quad \varepsilon \to 0. &(6.5)
\end{aligned}$$

Recall (5.10) and we have

$$\begin{aligned}
H_{-1} + H_{-2} &= \frac{Re(\alpha^l(1-\alpha))}{Re(1-\alpha)} \\
&\approx Re(\alpha^l(1+i)v)/v \approx Re(e^{-\nu x(1+i)}(1+i)) \quad as \quad \varepsilon \to 0.
\end{aligned}$$

Thus we get

$$I_1 \approx f(0)e^{-\nu x}(\cos(\nu x) + \sin(\nu x)) \quad as \quad \varepsilon \to 0.$$

On the other hand, we obtain

$$\begin{aligned}
H_{-1} - H_{-2} &= Im(\alpha^{l+1}\frac{\alpha}{1+\alpha})/Im(\frac{\alpha}{1+\alpha}) \\
&\approx Im(\alpha^{l+1}\frac{1}{2})/(-\frac{v}{4}) \approx -2Im(e^{-\nu x(1+i)})/v \quad as \quad \varepsilon \to 0.
\end{aligned}$$

Thus it is implied that

$$I_2 \approx f'(0)e^{-\nu x}\sin(\nu x))/\nu.$$

Therefore we have proved the following theorem by the new method, that is the scaling limit of the Markov chain $\{X_n\}$.

**Theorem 6.1** (K.Nishioka)**.** *For every differentiable function $f$, we have*

$$Q_x[e^{-\lambda\sigma}f(Z_\sigma)] = f(0)e^{-\nu x}(\cos(\nu x)+\sin(\nu x)) + \frac{f'(0)}{\nu}e^{-\nu x}\sin(\nu x), \tag{6.6}$$

*where $\nu = (2\lambda)^{1/4}$. In particular, it holds that*

$$Q_x[e^{i\mu Z_\sigma - \lambda\sigma}] = e^{-\nu x}(\cos(\nu x)+\sin(\nu x)) + \frac{i\mu}{\nu}e^{-\nu x}\sin(\nu x). \tag{6.7}$$

**Remark 2.** Let $\lambda \to 0$ in (6.5), and we obtain

$$u(x) = Q_x[f(Z_\sigma)] = f(0) + f'(0)x$$

which satisfies

$$u'''' = 0, \quad u(0) = f(0), \quad u'(0) = f'(0).$$

Here we find the differentiation of $f(x)$. This fact, which is due to K.Nishioka, seems to be remarkable for many probabilists.

As was shown in [N1], we can write the Laplace inverse of the above formula in an explicit form. From the Laplace inversion formula, a direct calculation shows that

$$\mathscr{L}^{-1}(e^{-\nu(a+ib)}) = \frac{i}{4\pi}\int_0^\infty u^3 e^{-\frac{u^4 t}{8}}(e^{-\frac{u}{2}(a-b+i(a+b))} - e^{-\frac{u}{2}(a+b-i(a-b))})du \tag{6.8}$$

when $|b| \le a$. Then we easily obtain the following functions

$$K(t,x) \equiv \mathscr{L}^{-1}(Re(e^{-\nu(1+i)x}(1+i)) \tag{6.9}$$

$$= \frac{1}{4\pi}\int_0^\infty u^3 e^{-u^4 t/8}(\sin xu - \cos xu + e^{-xu})du, \tag{6.10}$$

$$J(t,x) \equiv \mathscr{L}^{-1}(\frac{1}{\nu}e^{-\nu x}\sin\nu x) \tag{6.11}$$

$$= \frac{1}{4\pi}\int_0^\infty u^2 e^{-u^4 t/8}(\sin xu - \cos xu + e^{-xu})du, \tag{6.12}$$

which give us the density function of the first hitting time $\sigma$ and the hitting place:

$$Q_x[\sigma \in dt, Z_\sigma \in dy] = K(t,x)dt\delta(dy) - J(t,x)dt\delta'(dy). \tag{6.13}$$

## 7. Scaling limit of the transition probability with killing

In the previous section, we obtained $\tilde{g}_1(l,k)$. Here, we consider the scaling limit of $\tilde{g}_s(l,k)$, which is the Laplace transform of the transition probability of the biharmonic pseudo process with the absorbing boudary . We can find the same result with that in Nishioka[N1, section 9.1].

Let $l\varepsilon = x, k\varepsilon = y$ and $x,y$ be positive . Noting that $\Delta t = \varepsilon^4$, we consider

$$Q_x[\int_0^\sigma e^{-\lambda t}f(Z_t)dt] \stackrel{\text{def}}{=} \lim_{\varepsilon\to 0}\sum_{n=0}^\infty E_l[e^{-\lambda\varepsilon^4 n}f(\varepsilon X_n); n \le \sigma]\varepsilon^4. \tag{7.1}$$

Hence

$$\begin{aligned}&Q_x[\int_0^\sigma e^{-\lambda t}\delta_y(Z_t)dt]\\ \approx&\sum_{n=0}^\infty E_l[e^{-\lambda\varepsilon^4 n}1_k(X_n);n\le\sigma]\varepsilon^3\\ =&\tilde{g}_s(l,k)\varepsilon^3=(c_{k-l}-H_{-1}(s,l)c_{k+1}-H_{-2}(s,l)c_{k+2})\varepsilon^3,\end{aligned}$$

where $s=e^{-\lambda\varepsilon^4}$. On the other hand, since $-k-1<0<l$, we have

$$\begin{aligned}&p_n(l,-k-1)=P_l(X_n=-k-1)=P_l(X_n=-k-1;\sigma<n)\\ =&\sum_{j=-1,-2}\sum_{r=0}^n P_l[\sigma=r;X_\sigma=j]P_j[X_{n-r}=-k-1].\end{aligned}$$

Multiply the both side $s^n$, and sum in $n$, we obtain

$$c_{l+k+1}=\sum_{j=-1,-2}H_j(s,l)c_{k+1+j}.$$

Therefore

$$\tilde{g}_s(l,k)+c_{l+k+1}=c_{k-l}+R,$$

where

$$\begin{aligned}R=&\sum_{j=-1,-2}H_j(s,l)(c_{k+1+j}-c_{k-j})\\ =&\frac{H_{-1}+H_{-2}}{2}(c_{k-1}+c_k-(c_{k+1}+c_{k+2}))-\frac{H_{-1}-H_{-2}}{2}(c_{k-1}-c_k+c_{k+1}-c_{k+2}).\end{aligned}$$

From (5.16) ,(5.17) and the equalities following them, we can easily get

$$sv^4R\approx-4v^2Re(\alpha^{l+1}(1+i))Im(\alpha^k)-4vIm(\alpha^{l+1})Im(\alpha^k)$$

as $s\to1(v\to0)$. Then we obtain

$$\begin{aligned}\varepsilon^3R\approx&-\frac{4}{v^3}Im(\alpha^{l+1})Im(\alpha^k)\\ \approx&-\frac{4}{v^3}Im(e^{-(1+i)vx})Im(e^{-(1+i)vy})=\frac{4}{v^3}e^{-vx}\sin(vx)\sin(vy)\quad as\quad\varepsilon\to0.\end{aligned}$$

By remembering (5.10) and (6.4), it is easily checked that

$$\varepsilon^3c_{k-l}\approx\frac{1}{v^3}Re(e^{-(1+i)v|x-y|}(1+i)).$$

We may therefore see that

$$\int_0^\infty e^{-\lambda t}q(t,x)dt=\frac{1}{v^3}Re(e^{-(1+i)v|x|}(1+i)).$$

Thus, we conclude

$$\tilde{q}(t,x,y) \equiv q(t,x,y;t<\sigma) = q(t,x-y) - q(t,x+y) - \mathscr{L}^{-1}(\frac{4}{\nu^3}e^{-\nu(x+y)}\sin\nu x\sin\nu y).$$

Since

$$\mathscr{L}(\frac{\partial}{\partial x}q(t,x)) = -\frac{2}{\nu^2}e^{-\nu x}\sin\nu x$$

and (6.10) holds, we obtain the following result from the scaling limit of $\{X_n\}$.

**Theorem 7.1** (K.Nishioka)**.**

$$\tilde{q}(t,x,y) = q(t,x-y) - q(t,x+y) + 2J(t,y) * \frac{\partial}{\partial x}q(t,x).$$

**Remark 3.** The third term in the right hand of this equation shows an effect from the differential term of the first hitting place. For the Brownian motion, this term does not exist and the reflection principle holds.

From the above, we also get

$$\begin{aligned}
&\lim_{\lambda\to 0}\int_0^\infty e^{-\lambda t}\tilde{q}(t,x,y)dt \\
&= \lim_{\lambda\to 0}\frac{1}{\nu^3}(Re(e^{-(1+i)\nu|x-y|}(1+i)) - Re(e^{-(1+i)\nu(x+y)}(1+i)) - 4e^{-\nu(x+y)}\sin(\nu x)\sin(\nu y)) \\
&= \begin{cases} 4x^2y - \frac{4}{3}x^3 & (y \ge x) \\ 4xy^2 - \frac{4}{3}y^3 & (y \le x) \end{cases},
\end{aligned}$$

which is the scaling limit of $\tilde{g}_1(l,k)$. On the other hand, for the Brownian motion, it is well known that

$$\begin{aligned}
\int_0^\infty \tilde{q}(t,x,y)dt &= x+y-|x-y| \\
&= \begin{cases} 2x & (y \ge x) \\ 2y & (y \le x) \end{cases}.
\end{aligned}$$

## 8. Hitting measure to the boundary of finite intervals

In this section, we consider the first hitting time

$$\sigma = min\{n; X_n < 0 \quad or \quad X_n > L\}.$$

By an analogous argument as in section 4, we obtain

$$\tilde{g}(s,l,k) = c_{k-l} - \sum_{j=-1,-2,L+1,L+2} H_j(s,l)c_{k-j} \tag{8.1}$$

and

$$\begin{pmatrix} c_0 & c_1 & c_{L+2} & c_{L+3} \\ c_1 & c_0 & c_{L+3} & c_{L+4} \\ c_{L+2} & c_{L+3} & c_0 & c_1 \\ c_{L+3} & c_{L+4} & c_1 & c_0 \end{pmatrix} \begin{pmatrix} H_{-1} \\ H_{-2} \\ H_{L+1} \\ H_{L+2} \end{pmatrix} = \begin{pmatrix} c_{l+1} \\ c_{l+2} \\ c_{L+1-l} \\ c_{L+2-l} \end{pmatrix} \tag{8.2}$$

We consider the new variables

$$x=\frac{H_{-1}+H_{-2}+H_{L+1}+H_{L+2}}{2},y=\frac{H_{-1}-H_{-2}+H_{L+1}-H_{L+2}}{2}$$
$$u=\frac{H_{-1}+H_{-2}-H_{L+1}-H_{L+2}}{2},w=\frac{H_{-1}-H_{-2}-H_{L+1}+H_{L+2}}{2}.$$

Then we get a simultaneous equation

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \tag{8.3}$$
$$\begin{pmatrix} d & -c \\ -c & b \end{pmatrix}\begin{pmatrix} u \\ w \end{pmatrix}=\begin{pmatrix} h_3 \\ h_4 \end{pmatrix}, \tag{8.4}$$

where

$$\begin{aligned}
a &= 2(c_0+c_1)+c_{L+2}+2c_{L+3}+c_{L+4} \\
&= 2(1-Re(\alpha))+Re(\alpha^{L+2}(1-\alpha^2)), \\
b &= 2(c_0-c_1)+c_{L+2}-2c_{L+3}+c_{L+4} \\
&= -4v^2Im(\frac{\alpha}{1+\alpha})-2v^2Im(\frac{\alpha^{L+3}(1-\alpha)}{1+\alpha}), \\
d &= 2(c_0+c_1)-c_{L+2}-2c_{L+3}-c_{L+4} \\
&= 2(1-Re(\alpha))-Re(\alpha^{L+2}(1-\alpha^2)), \\
c &= c_{L+2}-c_{L+4} \\
&= -2v^2Im(\alpha^{L+3}), \\
h_1 &= c_{l+1}+c_{l+2}+c_{L+1-l}+c_{L+2-l}, \\
h_2 &= c_{l+1}-c_{l+2}+c_{L+1-l}-c_{L+2-l}, \\
h_3 &= c_{l+1}+c_{l+2}-c_{L+1-l}-c_{L+2-l}, \\
h_4 &= c_{l+1}-c_{l+2}-c_{L+1-l}+c_{L+2-l}.
\end{aligned}$$

We set

$$\gamma=Re(\alpha^{L+2}),\quad \rho=Im(\alpha^{L+2}).$$

Using (5.10),(5.13),(5.16) and (5.18), we obtain

$$\begin{aligned}
a &= 2v+Re(\alpha^{L+2}2(1+i)v)+o(v) \\
&= 2v(1+\gamma-\rho)+o(v), \\
b &= v^3-2v^2Im(\alpha^{L+2}\frac{1+i}{2}v)+o(v^3) \\
&= v^3(1-\gamma-\rho)+o(v^3), \\
d &= 2v(1-\gamma+\rho)+o(v), \\
c &= -2v^2\rho+o(v^2).
\end{aligned}$$

Thus we get

$$\begin{aligned}
det1 &\stackrel{\text{def}}{=} ab-c^2=2v^4(1-2\rho-\gamma^2-\rho^2)+o(v^4), \\
det2 &\stackrel{\text{def}}{=} db-c^2=2v^4(1+2\rho-\gamma^2-\rho^2)+o(v^4).
\end{aligned}$$

Moreover

$$
\begin{aligned}
h_1 &= Re((1-\alpha)(\alpha^{l+1}+\alpha^{L+1-l})) \approx vRe((1+i)(\alpha^{l+1}+\alpha^{L+1-l}))+o(v),\\
h_2 &= -2v^2 Im(\frac{1}{1+\alpha}(\alpha^{l+2}+\alpha^{L+2-l})) \approx -v^2 Im(\alpha^{l+2}+\alpha^{L+2-l})+o(v^2),\\
h_3 &= vRe((1+i)(\alpha^{l+1}-\alpha^{L+1-l}))+o(v),\\
h_4 &= -v^2 Im(\alpha^{l+2}-\alpha^{L+2-l})+o(v^2).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
x &= (b*h_1-c*h_2)/det1\\
&= \frac{Re((1+i)(\alpha^{l+1}+\alpha^{L+1-l}))(1-\gamma-\rho)-Im(\alpha^{l+2}+\alpha^{L+2-l})2\rho}{2(1-2\rho-\gamma^2-\rho^2)}+o(v),\\
y &= (-c*h_1+a*h_2)/det1\\
&= \frac{Re((1+i)(\alpha^{l+1}+\alpha^{L+1-l}))2\rho-Im(\alpha^{l+2}+\alpha^{L+2-l})2(1+\gamma-\rho)}{2v(1-2\rho-\gamma^2-\rho^2)}+o(\frac{1}{v}),\\
u &= (b*h_3+c*h_4)/det2\\
&= \frac{Re((1+i)(\alpha^{l+1}-\alpha^{L+1-l}))(1-\gamma-\rho)+Im(\alpha^{l+2}-\alpha^{L+2-l})2\rho}{2(1-\gamma^2+2\rho-\rho^2)}+o(v),\\
w &= (c*h_3+d*h_4)/det2\\
&= \frac{-2\rho Re((1+i)(\alpha^{l+1}-\alpha^{L+1-l}))+Im(\alpha^{l+2}-\alpha^{L+2-l})2(1-\gamma+\rho)}{2v(1-\gamma^2+2\rho-\rho^2)}+o(\frac{1}{v}).
\end{aligned}
$$

Now we consider the scaling limit. We set

$$x = l\varepsilon, \quad a = L\varepsilon, \quad v = \varepsilon\nu$$

where $\nu = (2\lambda)^{1/4}$. For the pseudo biharmonic process $\{Z_t\}$, we consider

$$\sigma = inf\{t; Z_t < 0 \quad or \quad Z_t > a\}.$$

Since we approximate the pseudo process by the scaling limit of $\{X_n\}$, we define

$$Q_x[e^{-\lambda\sigma} f(Z_\sigma)] \overset{\text{def}}{=} \lim_{\varepsilon\to 0} E_l[e^{-\varepsilon^4\lambda\sigma} f(\varepsilon X_\sigma)], \tag{8.5}$$

where $x = l\varepsilon$ and $a = L\varepsilon$ are fixed and $f$ is a bounded $C^1$-function. We shall evaluate

$E_l[e^{-\varepsilon^4\lambda\sigma} f(\varepsilon X_\sigma)]$:

$$\begin{aligned}
&E_l[e^{-\varepsilon^4\lambda\sigma} f(\varepsilon X_\sigma)] \\
&= f(-\varepsilon)H_{-1}(v,l) + f(-2\varepsilon)H_{-2}(v,l) + f(a+\varepsilon)H_{L+1}(v,l) + f(a+2\varepsilon)H_{L+2}(v,l) \\
&= f(-\varepsilon)\frac{x+y+u+w}{2} + f(-2\varepsilon)\frac{x-y+u-w}{2} \\
&\quad + f(a+\varepsilon)\frac{x+y-u-w}{2} + f(a+2\varepsilon)\frac{x-y-u+w}{2} \\
&= (f(-\varepsilon)+f(-2\varepsilon)+f(a+\varepsilon)+f(a+2\varepsilon))\frac{x}{2} + (f(-\varepsilon)-f(-2\varepsilon)+f(a+\varepsilon)-f(a+2\varepsilon))\frac{y}{2} \\
&\quad + (f(-\varepsilon)+f(-2\varepsilon)-f(a+\varepsilon)-f(a+2\varepsilon))\frac{u}{2} + (f(-\varepsilon)-f(-2\varepsilon)-f(a+\varepsilon)+f(a+2\varepsilon))\frac{w}{2} \\
&\approx (f(0)+f(a))x + \varepsilon(f'(0)-f'(a))\frac{y}{2} + (f(0)-f(a))u + \varepsilon(f'(0)+f'(a))\frac{w}{2}
\end{aligned}$$

where we set

$$s = e^{-\varepsilon^4\lambda}.$$

We define two new functions

$$C(x) = e^{-\nu x}\cos(\nu x), \quad S(x) = e^{-\nu x}\sin(\nu x). \tag{8.6}$$

It is immediate to check that those functions satisfy

$$g'''' = -4\nu^4 g = -8\lambda g.$$

Since

$$\begin{aligned}
\alpha^{L+2} &\approx e^{-(1+i)\nu\varepsilon(L+2)} \approx e^{-(1+i)\nu a}, \\
\alpha^{l+1} &\approx e^{-(1+i)\nu x}, \\
\alpha^{L+1-l} &\approx e^{-(1+i)\nu(a-x)},
\end{aligned}$$

we get

$$\begin{aligned}
\gamma &\approx C(a), \quad \rho \approx -S(a), \\
Re((1+i)(\alpha^{l+1}+\alpha^{L+1-l})) &\approx C(x)+S(x)+C(a-x)+S(a-x), \\
Im(\alpha^{l+2}+\alpha^{L+2-l}) &\approx -S(x)-S(a-x), \\
&\cdots\cdots.
\end{aligned}$$

Finally, we obtain the following theorem:

**Theorem 8.1.** *Let $f$ be a bounded function in $C^1$, and set*

$$
\begin{aligned}
& w(\lambda,x) \stackrel{\mathrm{def}}{=} Q_x[e^{-\lambda\sigma} f(Z_\sigma)] \\
= & (f(0)+f(a))\frac{(C(x)+S(x)+C(a-x)+S(a-x))(1-C(a)+S(a))-2(S(x)+S(a-x))S(a)}{2(1+2S(a)-C^2(a)-S^2(a))} \\
& +(f(0)-f(a))\frac{(C(x)+S(x)-C(a-x)-S(a-x))(1+C(a)-S(a))+2(S(x)-S(a-x))S(a)}{2(1-2S(a)-C^2(a)-S^2(a))} \\
& +(f'(0)-f'(a))\frac{-(C(x)+S(x)+C(a-x)+S(a-x))S(a)+(S(x)+S(a-x))(1+C(a)+S(a))}{2\nu(1+2S(a)-C^2(a)-S^2(a))} \\
& +(f'(0)+f'(a))\frac{(C(x)+S(x)-C(a-x)-S(a-x))S(a)+(S(x)-S(a-x))(1-C(a)-S(a))}{2\nu(1-2S(a)-C^2(a)-S^2(a))}.
\end{aligned}
$$

*Then $w(\lambda,x)$ satisfies*

$$
-\frac{1}{8}\frac{\partial^4 w}{\partial x^4}-\lambda w=0
$$

*and*

$$
w(\lambda,0)=f(0), \quad w(\lambda,a)=f(a)
$$

$$
\frac{\partial w}{\partial x}(\lambda,0)=f'(0), \qquad \frac{\partial w}{\partial x}(\lambda,a)=f'(a).
$$

**Remark 4.**

(i) As $\lambda$ goes to zero, we obtain

$$
\begin{aligned}
u(x) & \stackrel{\mathrm{def}}{=} Q_x[f(Z_\sigma)] \\
& = \frac{f(0)+f(a)}{2}+\frac{f(0)-f(a)}{2}(1+4(\frac{x}{a})^3-6(\frac{x}{a})^2) \\
& \quad +\frac{a(f'(0)-f'(a))}{2}(-(\frac{x}{a})^2+\frac{x}{a}) \\
& \quad +\frac{a(f'(0)+f'(a))}{2}(2(\frac{x}{a})^3-3(\frac{x}{a})^2+\frac{x}{a})
\end{aligned}
$$

which satisfies

$$
u''''=0, \quad u(0)=f(0), \quad u(a)=f(a), \quad u'(0)=f'(0), \quad u'(a)=f'(a).
$$

(ii) When $a$ goes to infinity, for any fixed values $f(a)$ and $f'(a)$, we obtain

$$
Q_x[e^{-\lambda\sigma} f(Z_\sigma)] = f(0)(C(x)+S(x))+\frac{f'(0)}{\nu}S(x)
$$

which naturally coincides with Theorem 6.1.

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DEPARTMENT OF NATURAL SCIENCES
TOKYO DENKI UNIVERSITY
HATOYAMA-MACHI,
SAITAMA HIKI, 350-0394,
JAPAN

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